Covering invariant categories and noncommutative supergeometry

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(arXiv:1904.13130)

Definition

A category $\mathfrak{C}$ is said to be *spatial* if there is a *spatial functor* $\mathfrak{Sp} : \mathfrak{C} \to \mathfrak{Top}$ from $\mathfrak{C}$ to the category $\mathfrak{Top}$ of topological spaces.

Theorem

(Gelfand-Naĭmark). Let A be a commutative C^*-algebra and let $\mathcal{X}$ be the spectrum of A. There is the natural $*$-isomorphism $\gamma : A \to C_0(\mathcal{X})$.

Definition

A species of sets $\mathfrak{NC}$ (à la Bourbaki) which for any $X \in \mathfrak{C}$ naturally defines an operator space $\mathfrak{OSp}\,(X)$ is said to be a *noncommutative spatial category*.

Since $C_0\,(\mathcal{X})$ is an operator space any spatial category is a noncommutative spatial category.

Definition

A spatial category $\mathfrak{C}$ is said to *topologically covering invariant* if for any object X of $\mathfrak{C}$ and for any covering $p : \widetilde{\mathcal{X}} \to \mathfrak{Sp}\left(X\right)$ there is the object $\widetilde{X}$ of the category $\mathfrak{C}$ such that $\mathfrak{Sp}\left(X\right) \cong \widetilde{\mathcal{X}}$ and the morphism $\pi : \widetilde{X} \to X$ such that $\mathfrak{Sp}\left(\pi\right) \cong p$.

There are noncommutative coverings of operator spaces such that topological coverings correspond to noncommutative ones.

Definition

A noncommutative spatial category $\mathfrak{NC}$ is said to *algebraically covering invariant* if for any object X of $\mathfrak{NC}$ and for any noncommutative covering $\widetilde{Y} \to \mathfrak{OSp}\left(X\right)$ there is the object $\widetilde{X}$ of the category $\mathfrak{NC}$ such that $\mathfrak{OSp}\left(\widetilde{X}\right) \cong \widetilde{Y}$.

Any topologically covering invariant category is algebraically covering invariant.

Definition

(P. Ivankov) An injective *-homomorphism $A \hookrightarrow \widetilde{A}$ of unital C^*-algebras is an *unital noncommutative finite-fold covering* if following conditions hold: 1) $\widetilde{A}$ is a finitely generated left and right A-module, 2) if $\mathrm{Aut}\left(\widetilde{A}\right)$ is a group of *-automorphisms of $\widetilde{A}$ then the group $G = \left\{ g \in \mathrm{Aut}\left(\widetilde{A}\right) \;\middle|\; ga = a; \quad \forall a \in A \right\}$ is finite, 3) $A \cong \widetilde{A}^G \stackrel{\text{def}}{=} \left\{ a \in \widetilde{A} \;\middle|\; a = ga; \; \forall g \in G \right\}$.

Theorem

(A. Pavlov, E. Troitsky, P. Ivankov) If $\mathcal{X}$ is compact Hausdorff space then there is the 1-1 map between finite-fold coverings of $\mathcal{X}$ and unital noncommutative finite-fold coverings of $C\left(\mathcal{X}\right)$.

Definition

A *superspace* is a pair $(M, \mathcal{O}_M)$ which contains a topological space M and a sheaf of supercommutative rings $\mathcal{O}$ on M such that for all $x \in M$ the germ $\mathcal{O}_x = \mathcal{O}_{M,x}$ is a local ring.

Definition

A *morphism of superspaces* $(M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ is a pair (φ, ψ) where $\varphi : M \to N$ is a continuous map, $\psi : \mathcal{O}_N \to \varphi_* (\mathcal{O}_M)$ is a morphism of sheaves of rings such that for any $x \in M$ the homomorphism $\psi_x : \mathcal{O}_{N,\varphi(x)}$ is local, i.e: $\psi_x \left(\mathfrak{m}_{\varphi(x)}\right) = \mathfrak{m}_x$

Above definitions yield a spatial category $\mathfrak{Ss}$ of superspaces where the spatial functor $\mathfrak{Sp}$ is the forgetful functor $(M, \mathcal{O}_M) \mapsto M$.

Lemma

A category $\mathfrak{Ss}$ of superspaces is topologically covering invariant.

Proof.

Let $(M, \mathcal{O}_M)$ be an object of the category $\mathfrak{Ss}$, and let $p : \widetilde{M} \to M$ be a covering. If $\mathcal{O}_{\widetilde{M}} \stackrel{\text{def}}{=} p^{-1}(\mathcal{O}_M)$ is the inverse image of the sheaf then $\mathcal{O}_{\widetilde{M}}$ is a sheaf of rings which satisfies to the definition of the superspace, i.e. the pair $\left(\widetilde{M}, \mathcal{O}_{\widetilde{M}}\right)$ is a suprespace. It is easy to construct the morphism of sheaves $\psi : \mathcal{O}_{\widetilde{M}} \to \varphi_* (\mathcal{O}_M)$ which satisfies the definition of morphism of superspaces. ∎

In the above situation the forgetful functor $\mathfrak{Sp}$ has a very coarse information. Here we construct a more informative map $\mathfrak{OSp}$ such that

- $\mathfrak{OSp}\left(\left(M, \mathcal{O}_M\right)\right)$ is an operator space.
- If there is a noncommutative covering of $\mathfrak{OSp}\left(\left(M, \mathcal{O}_M\right)\right)$ by the operator space $\widetilde{X}$ then there is the superspace $\left(\widetilde{M}, \mathcal{O}_{\widetilde{M}}\right)$ such that $\mathfrak{OSp}\left(\left(\widetilde{M}, \mathcal{O}_{\widetilde{M}}\right)\right) = \widetilde{X}$, i.e.

It means that the category $\mathfrak{Ss}$ of superspaces is algebraically covering invariant.

The C^*-algebra $C(M)$ of complex-valued continuous functions yields a presheaf on M. Denote by $\mathcal{C}$ the associated sheaf. Similarly one has the sheaves $\mathcal{C}^\infty$ and $\mathcal{C}^\omega$ of smooth and analytical functions respectively. Denote by $\mathcal{O}^{\mathcal{C}}_M \overset{\mathrm{def}}{=} \mathcal{O}_M \otimes_{\mathcal{C}^\infty} \mathcal{C}$ (resp. $\mathcal{O}^{\mathcal{C}}_M \overset{\mathrm{def}}{=} \mathcal{O}_M \otimes_{\mathcal{C}^\omega} \mathcal{C}$) if $\mathcal{O}_M$ is smooth (resp. analytical). The sheaf $\mathcal{O}^{\mathcal{C}}_M$ corresponds to the bundle of finite dimensional complex algebras. It turns out that there is the inclusion

$$\Gamma\left(M, \mathcal{O}^{\mathcal{C}}_M\right) \hookrightarrow A$$

where A is a homogeneous of order n C^*-algebra, i.e. corresponds to a bundle over M with the fiber $\mathbb{M}_n(\mathbb{C})$. The C^*-norm completion X of $\Gamma\left(M, \mathcal{O}^{\mathcal{C}}_M\right)$ in A is an operator algebra which is a special case of an operator space. Denote by $\mathfrak{OSp}\left(\left(M, \mathcal{O}_M\right)\right) \overset{\mathrm{def}}{=} X$.

Theorem

(P. Ivankov) The category $\mathfrak{Ss}$ of superspaces is algebraically covering invariant.

Proof.

In arXiv:1904.13130 the theory of coverings of operator spaces is developed. In particular coverings of operator spaces which are subspaces of a homogeneous of order n C^*-algebra are defined. The theorem follows from these calculations. ∎

Supersymmetric quantum theory and (non-commutative) differential geometry. by J. Fröhlich, O. Grandjean, A. Recknagel.

Definition

A quadruple $(\mathcal{A}, \mathcal{H}, D, \gamma)$ will be called a set of $N = 1$ *(even) spectral data* if

1. $\mathcal{H}$ is a separable Hilbert space;
2. $\mathcal{A}$ is a unital *-algebra acting faithfully on $\mathcal{H}$ by bounded operators;
3. D is a self-adjoint operator on $\mathcal{H}$ such that
   - for each $a \in \mathcal{A}$, the commutator $[D, a]$ defines a bounded operator on $\mathcal{H}$,
   - the operator $\exp(-\varepsilon D^2)$ is trace class for all $\varepsilon > 0$;
4. γ is a $\mathbb{Z}_2$-grading on $\mathcal{H}$, i.e., $\gamma = \gamma^* = \gamma^{-1}$, such that

$$\{\gamma, D\} = 0, \quad [\gamma, a] = 0 \quad \forall a \in \mathcal{A}. \tag{1}$$

Indeed $N=1$ (even) spectral data should satisfy to a set axioms so it a species of sets à la Bourbaki which we denote by $\mathfrak{N}_1$. The algebra $\mathcal{A}$ plays the role of the "algebra of smooth functions over a non-commutative space" The given by (1) equations

$$\{\gamma, D\} = 0, \quad [\gamma, a] = 0 \quad \forall a \in \mathcal{A}.$$

are equations of superalgebra. If we denote by A the C^*-norm completion of $\mathcal{A}$ then $\mathfrak{N}_1$ becomes a noncommutative spatial category such that $\mathfrak{O}\mathfrak{S}\mathfrak{p}\left(\mathcal{A}, \mathcal{H}, D, \gamma\right) \stackrel{\text{def}}{=} A$.

Theorem

(P. Ivankov) The noncommutative spatial category $\mathfrak{N}_1$ is algebraically covering invariant. (See arXiv:1904.13130).

Any Riemannian manifold M with a spin$^{\mathrm{c}}$ bundle S yields $N=1$ (even) spectral data $\left(C^{\infty}\left(M\right), L^2\left(M,S\right), \not{D}, \gamma\right)$. We say that $\left(C^{\infty}\left(M\right), L^2\left(M,S\right), \not{D}, \gamma\right)$ is *commutative spectral triple*. The species of sets $\mathfrak{CM}_1$ of commutative spectral triple is a spatial category such that $\mathfrak{Sp}\left(\left(C^{\infty}\left(M\right), L^2\left(M,S\right), \not{D}, \gamma\right)\right) \stackrel{\mathrm{def}}{=} M$. Here we give the purely geometric version of the above theorem.

Theorem

(P. Ivankov) The noncommutative spatial category $\mathfrak{CM}_1$ is covering invariant.

Proof.

If $p: \widetilde{M} \to M$ is a covering then there is the natural structure of Riemannian manifold on $\widetilde{M}$. If $\widetilde{S}$ is the p-inverse image of S then $\widetilde{S}$ is a spin$^{\mathrm{c}}$ bundle on $\widetilde{M}$. So there is a commutative spectral triple $\left(C^{\infty}\left(\widetilde{M}\right), L^2\left(\widetilde{M},\widetilde{S}\right), \widetilde{\not{D}}, \widetilde{\gamma}\right)$. ∎

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$$
\begin{array}{ccccc}
 & & N = 1 & : & \text{Connes' complex } \Omega_D \\
 & & \twoheaduparrow\downarrow & & \\
N = 2 & \leftrightarrows & N = (1,1) & : & \text{de Rham complex} \\
\twoheaduparrow & & \twoheaduparrow & & \\
N = 4^+ & \leftrightarrows & N = (2,2) & : & \text{Dolbeault complex} \\
\twoheaduparrow & & \twoheaduparrow & & \\
N = 8 & \leftrightarrows & N = (4,4) & : & \text{Hyperkähler geometry}
\end{array}
$$

Table: Noncommutative supresymmetric geometries*

Type of supersymmetry	Data set	Noncommutative samples
$N=1$	$(\mathcal{A},\mathcal{H},D,\gamma)$	Yes
$N=(1,1)$	$(\mathcal{A},\mathcal{H},\mathtt{d},\gamma,*)$	Not known
$N=(2,2)$	$(\mathcal{A},\mathcal{H},\partial,\overline{\partial},\gamma,*)$	Not known
$N=(4,4)$	$(\mathcal{A},\mathcal{H},G^{a\pm},\overline{G}^{a\pm},T^i,\overline{T}^i,\gamma,*)$	Not known

* Supersymmetric quantum theory and (non-commutative) differential geometry. by J. Fröhlich, O. Grandjean, A. Recknagel.

Expect further results.
I appreciate if you join this promising research.

Thank You!