Noncommutative Instanton

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Abstract

Noncommutative geometry is analogue of commutative one. However it contains new types of geometrical objects. So provides new physical ideas and these ideas can be analogue of well known objects. But noncommutative objects have essentially new properties. For example topological properties of ordinary (commutative) sphere S^3 provides instanton solution. Noncommutative sphere provides analogue of ordinary instanton. However this instanton has essentially new properties. This article is devoted to discussion of these properties.

1 Introduction

A set of noncommutative analogues of ordinary geometrical objects has been discovered during last decades. One of them is noncommutative sphere [1]. Usual 3D sphere may be considered as a topological space. However there exists another algebraic representation where topological space is replaced by commutative algebra of continuous complex valued functions on the space [5]. There is one to one correspondence between objects of these representations. In noncommutative geometry commutative algebra is replaced by noncommutative one. Since commutative algebra is a particular case of noncommutative one usual (commutative) geometry could be considered as a case of noncommutative geometry. We would like to find a noncommutative algebra that is very similar to algebra of continuous complex valued functions defined on 3D sphere. These algebra also should provide interesting physical models. Survey of such models could be found at [6]. If noncommutative algebra is compatible with structure of spectral triple [7] then structure of differential forms, curvature and other features of classical geometry may be constructed. Therefore this algebra may provide interesting physical model. Algebra of complex functions of 3D sphere could be generated by four real valued functions x_1,...,x_4 those satisfy to following equations:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1. \tag{1}$$

If we introduce complex valued functions $\alpha = x_1 + ix_2$, $\beta = x_3 + ix_4$ then when we can replace (1) by the following equation:

$$\alpha\alpha^* + \beta\beta^* = 1. \tag{2}$$

Noncommutative analogue if this algebra [1] is is generated by two elements α, β and satisfies to following relations.

$$\begin{array}{rcl} \alpha^*\alpha + \beta^*\beta = I, & & \alpha\alpha^* + q^2\beta\beta^* = I, \\ \alpha\beta - q\beta\alpha = 0, & & \alpha\beta^* - q\beta^*\alpha = 0, \\ \beta^*\beta & = & \beta\beta^*. \end{array}$$

where q is a real number and $0 < q \leq 1$.

This algebra is denoted by $C(SU_q(2))$. It is clear that if we suppose that $q = 1$ then this algebra is commutative and it satisfies to relations (1). If $q \approx 1$ then algebra $C(SU_q(2))$ could be considered as noncommutative approximation of algebra $C(S^3)$ of continuous complex valued functions on 3D sphere. $C(SU_q(2))$ admits the structure of spectral triple[7] and noncommutative case ($q < 1$) has nontrivial invariants those are trivial if $q = 1$. Although $C(SU_q(2)$ can be considered as approximation of $C(S^3)$ it has another values of global invariants. For example $K_1(C(SU_q(2)) = \mathbb{Z}$ but $K_1(C(\mathbb{S}^3)) = 0$. Noncommutative analogue of fundamental group [4] developed by Nikolay Ivankov in collaboration with author is trivial for $C(S^3)$. But I have some reasons to expect that fundamental group of $C(SU_q(2)$ is not trivial. Discussion about it you can find at [11]. Using this difference we can build new type of instanton.

2 Instanton in brief

Brief description of instanton you can find at [3]. Instanton is gauge field A on Rb^4. If we consider S^3 as infinite boundary of Rb^4 then field strength tensor equal to zero at S^3. Instanton is caused by existence of homotopically nontrivial continuous maps $S^3 \to G$ where G is gauge group of physical theory. Existence of such maps is possible if and only if

$$\pi_3(G) \neq 0$$

Gauge group of pure electromagnetism is $U(1)$. Since $\pi_3(G) = 0$ is trivial instanton pure electromagnetism does not allow instanton.

3 Noncommutative instanton

Group $U(1)$ has following nontrivial invariant $pi_1(U(1)) = \mathbb{Z}$. As it is expected $\pi_1(C(SU_q(2)) \neq 0$. So we expect pure electromagnetic instanton in noncommutative case. Here we try to construct it without notion of fundamental group. We will use K - theory only. First of all every continuous map $f : X \to U(1)$

from Hausdorff locally compact topological space X to $U(1)$ can be considered as unitary element of $u \in C(X)$. This map defines element of $[u] \in K_1(X)$. if $[u] \neq 0$ then f is not trivial. Using these properties we can construct analogue of instanton. According to [1] algebra $C(SU_q(2))$ contains such unitary element $u \in C(SU_q(2))$ that $[u] \neq 0$ in $K_1(C(SU_q(2))$. This element can be considered as homotopically nontrivial map from $SU_q(2)$ to $U(1)$. Using this element we construct instanton solution by spectral action principle [2] that is analogue of ordinary physical action. According to it gauge field A can be obtained from unitary u by the following way:

$$A = u[D, u^*];$$

Where D is Dirac operator [2].

These gauge field can be considered as noncommutative instanton. It should be make some remarks. Element u does not belong to smooth pre C^* - algebra of noncommutative sphere $SU_q(2)$. It means that commutator $[D, u^*]$ has no direct sense. However since smooth pre C^* - algebra is dense in its norm completion we can approximate u by smooth elements. Ordinary instanton is gauge field on whole $\mathbb{R}^4$. But now we have gauge field on $SU_q(2)$ that is analogue of S^3 only. We need noncommutative analogue of $\mathbb{R}^4$ and instanton solution on it.

4 Prospects

Instanton inspires a lot of physical ideas and hypothesis. One of them is existence of axion [12]. A lot of experiments can not approve these ideas or disprove them. Maybe new features of noncommutative instanton enable us to find experimentally proved predictions.

References

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