Coverings in Noncommutative Geometry

Petr R. Ivankov [1] Nikolay P.Ivankov [2]

[1]Avtomatika-S, Moscow, Russia

[2]Max-Planck-Institute for Mathematics, Bonn, Germany

A Spectral Triple was firstly introduced by Alain Connes as a generalization of Riemann manifold, or, in other words, a quantization of a Riemannian manifold. The idea is follows:

Having a Riemann manifold M, one may consider an algebra of smooth functions on M. This algebra is, obviously, commutative. Roughly speaking, Alain Connes suggested to consider a case when a commutative algebra is replaced by a noncommutative one, preserving the main properties of manifolds.

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But the crucial new "property" occurs from from this "quantization" conjecture. From the point of view of classical differential geometry every point could be identified with a nontrivial character on the algebra of functions and vise versa.

However, when algebra is noncommutative, there may be NO nontrivial characters. Thus, the notions of point, subset, neighborhood, locality and many others have in general no sense any more.

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Definition

A spectral triple is a set of five objects:

- $\mathcal{A}$ - a unital pre-C^*-algebra (an analogue of algebra of smooth functions on manifold)
- $\mathcal{H}$ - a Hilbert space, carrying a faithful representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ (an analogue of square integrable sections of spinor bundle)
- $\mathcal{D}$ - unbounded selfadjoint operator on $\mathcal{H}$ with compact resolvent (an analogue of Dirac operator)
- J - an antilinear isometry (needed almost only for noncommutative purpose)
- Γ - selfadjoint unitary operator on $\mathcal{H}$ (only for so-called even dimensional spectral triples)

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Objects mentioned above should satisfy following axioms

1. A spectral triple has an integer **classical dimension** n where dimension is defined by $n^{-1} = \sup_{\beta \in \mathbb{R}}(\beta\colon \mu_k = O(k^{-\beta}))$. Here μ_k is k'th eigenvalue of $|D + P|^{-1}$
2. A spectral triple is **real** i.e. for all $a, b \in \mathcal{A}$ one has $[a, b^\circ] := [a, Jb^* J^\dagger] = 0$ and the following commutation relations hold: $J^2 = \pm 1$, $J\mathcal{D} = \pm \mathcal{D}J$, $J\Gamma = \pm \Gamma J$.
3. An operator $\mathcal{D}$ is of **order one** i.e. $\forall a, b \in \mathcal{A}$ holds $[[D, a], b^\circ] = 0$ (noncommutative Leibnitz rule).
4. For all $a \in \mathcal{A}$ an operator $[D, a]$ could be extended to an element of $\mathcal{B}(\mathcal{H})$ and both a and $[D, a]$ belong to the domain of derivation $\delta(T) := [|D|, T]$.
5. There is a Hochshield cycle $c \in C_n(\mathcal{A}, \mathcal{A} \otimes \mathcal{A}^\circ)$ such that $\pi_D(c) = \Gamma$, if the dimension is even and $\pi_D(c) = 1$ otherwise.
6. The space $\mathcal{H}_\infty := \bigcap_{k=0}^{\infty} Dom(\mathcal{D}^k)$ is a finitely generated left $\mathcal{A}$-module (analogue of smooth sections of spinor bundle).
7. The index map of D determines a nondegenerate pairing on the K-theory of the algebra $\mathcal{A}$ (**Poincaré duality**).

Anyway, any object is not so interesting as a thing in itself. The natural way of development is to find the relations between objects of this kind, or, in other words, a "good" morphism, with which one can construct interesting functors and invariants.

The most common morphisms in today's noncommutative geometry are

- **Unitary equivalence** . Distinguishes spectral triples which are the same up to isometry.
- **Morita equivalence** . Distinguishes spectral triples with "same topological space".

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Still, in both algebraic topology and noncommutative geometry exists such an invariant as fundamental group, and since one consider NCG as a **geometry**, it would be nice to obtain a similar invariant.

Problem

There is no point in NCG, so one can not define this invariant with use of homotopic curves.

Clue

In algebraic geometry a fundamental group is introduced with the use of s.c. étale morphism, which is an analogue of unramified covering. This caused by the fact that the topology in algebraic geometry is non-Hausdorff.

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- one may define a **fundamental group** with use of **unramified coverings**,

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- the **unramifiedness** of the covering itself is provided by preservation of certain **differential structure given by a Dirac operator** $\mathcal{D}$ on a spectral triple.

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The way to construct a fundamental group is described in in an authors preprint. We shall mostly consider the notion of covering.

- First observe that a covering morphism of two topological spaces $X \to Y$ defines an inclusion $\mathcal{A}(Y) \hookrightarrow \mathcal{A}(X)$ of algebras of (continuous) functions on these spaces. Thus, to mimic it, for two given spectral triples $(\mathcal{A}, \mathcal{H}_{\mathcal{A}}, \mathcal{D}_{\mathcal{A}})$ and $(\mathcal{B}, \mathcal{H}_{\mathcal{B}}, \mathcal{D}_{\mathcal{B}})$ we make an inclusion morphism $f : \mathcal{B} \hookrightarrow \mathcal{A}$
- As it was stated above, a **unitary equivalence** defines an **"isometry"** on a spectral triple. Thus, for given spectral triple $\mathfrak{A} := (\mathcal{A}, \mathcal{H}_{\mathcal{A}}, \mathcal{D}_{\mathcal{A}})$ one may define an **isometry group** $\mathcal{G}(\mathfrak{A})$. Note: unitary equivalences are obtained with use of Dirac operator.

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- Now, consider the subgroup of all $\sigma \in \mathcal{G}(\mathfrak{A})$ such that $\sigma(b) = b$ for all $b \in Im(f)$. This group is called a **covering group** and denoted by $\mathcal{G}(\mathfrak{A}, \mathfrak{B})$. For now only finite covering groups are considered.
- Finally, one may define a projector $P_{\mathcal{G}(\mathfrak{A},\mathfrak{B})} = \frac{1}{|\mathcal{G}(\mathfrak{A},\mathfrak{B})|} \sum_{g \in \mathcal{G}(\mathfrak{A},\mathfrak{B})} g$,

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And, of course...

Axioms

1. $\mathfrak{A}$ and $\mathfrak{B}$ have the same classical dimension n.
2. $\mathcal{A}$ is a finitely generated projective $\mathcal{B}$-module.
3. The group $\mathcal{G}(\mathfrak{A}, \mathfrak{B})$ is finite and $P_{\mathcal{G}}(\mathfrak{A}, \mathfrak{B})(\mathcal{A}) = f(\mathcal{B})$
4. There is a surjective group homomorphism $f^* : \mathcal{G}(\mathfrak{A}) \to \mathcal{G}(\mathfrak{B})$, that $\forall \sigma \in \mathcal{G}(\mathfrak{A})$ and $\forall a \in \mathcal{A}, \forall b \in \mathcal{B}$ following relations hold: $P\sigma(a) = f^*(\sigma)P(a)$ and $ff^*(\sigma)(b) = \sigma f(b)$
5. Spaces $\mathcal{H}_{\mathcal{A}} \approx \mathcal{H}_{\mathcal{B}} \otimes \mathcal{A}$ are isomorphic as linear spaces.
6. Operators $\mathcal{D}_{\mathcal{B}}, J_{\mathcal{B}}\,(,\Gamma_{\mathcal{B}})$ are restrictions of $\mathcal{D}_{\mathcal{A}}, J_{\mathcal{A}}\,(,\Gamma_{\mathcal{A}})$ consequently.
7. An extension of an inclusion f maps fundamental Hochschield cycle of $\mathfrak{B}$ maps to fundamental Hochschield cycle on $\mathfrak{A}$ (volume form preservation).

- It was proved that the composition of two such morphisms is again a morphism, so spectral triples with covering morphisms may be considered as a **category**.
- The theory is nonempty. There exist several examples of finite coverings for classical spectral triples. For each spectral triple one may construct a fundamental group using the technique described in authors' preprint.
- Work is still in progress, but yet there was found a more general approach introduced by Bram Mesland, which contains the notion of covering as a particular case. Now authors are working on both more deep study of coverings and so-called "correspondences of proper KK-cycles".

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Thank You!