Noncommutative Generalization of Hurewicz homomorphism

Petr R. Ivankov [1]

[1]RPKB, Moscow, Russia

1. Let X be topological space, $x_0 \in X$ and $\pi_n(X, x_0)$, $H_n(X)$ are n-th ($n \in \mathbb{N}$) and homotopy group of singular homology then there is following homomorphism:
   $\phi : \pi_n(X, x_0) \to H_n(X)$.
   ϕ is named Hurewicz homomorphism.
   Pair (X, x_0) is named pointed space.
2. if $n = 1$ then homomorphism is defined by following way: Let S^1 be a circle then $H_1(S^1) \equiv \mathbb{Z}$. Let $c \in H_1(S^1)$ be generator of $H_1(S^1)$. Then Hurewicz homomorphism is defined by following expression:
   If $[f] \in [S^1, s_0] = \pi_1(X, x_0)$ then $\phi([f]) = H_1(f)(c)$.
3. Let us try generalize Hurewicz homomorphism for $n = 1$ to noncommutative case.

1. Let X be topological space, $x_0 \in X$ and $\pi_n(X, x_0)$, $H_n(X)$ are n-th ($n \in \mathbb{N}$) and homotopy group of singular homology then there is following homomorphism:
   $\phi : \pi_n(X, x_0) \to H_n(X)$.
   ϕ is named Hurewicz homomorphism.
   Pair (X, x_0) is named pointed space.
2. if $n = 1$ then homomorphism is defined by following way: Let S^1 be a circle then $H_1(S^1) \equiv \mathbb{Z}$. Let $c \in H_1(S^1)$ be generator of $H_1(S^1)$. Then Hurewicz homomorphism is defined by following expression:
   If $[f] \in [S^1, s_0] = \pi_1(X, x_0)$ then $\phi([f]) = H_1(f)(c)$.
3. Let us try generalize Hurewicz homomorphism for $n = 1$ to noncommutative case.

1. Let X be topological space, $x_0 \in X$ and $\pi_n(X, x_0)$, $H_n(X)$ are n-th ($n \in \mathbb{N}$) and homotopy group of singular homology then there is following homomorphism:
$\phi : \pi_n(X, x_0) \to H_n(X)$.
ϕ is named Hurewicz homomorphism.
Pair (X, x_0) is named pointed space.
2. if $n = 1$ then homomorphism is defined by following way: Let S^1 be a circle then $H_1(S^1) \equiv \mathbb{Z}$. Let $c \in H_1(S^1)$ be generator of $H_1(S^1)$. Then Hurewicz homomorphism is defined by following expression:
If $[f] \in [S^1, s_0] = \pi_1(X, x_0)$ then $\phi([f]) = H_1(f)(c)$.
3. Let us try generalize Hurewicz homomorphism for $n = 1$ to noncommutative case.

1. The Gelfand - Naimark theorem can be thought of as the construction of two contravariant functors (cofunctors for short) from the category of locally compact Hausdorff spaces to the category of C^*-algebras. The first cofunctor C takes a compact space X to the C^*-algebra $C(X)$ of continuous complex-valued functions on X, and takes a continuous map $f: X \to Y$ to its transpose $C(f): C(Y) \to C(X)$ defined by following way: $C(f) = (h \mapsto hf); (h \in C(Y))$.
2. Otherwise there exists inverse functor M that sets to any commutative C^*-algebra A space of its characters $M(A)$. Many topological results related to locally compact spaces has its (noncommutative) algebraic analogues.

1. The Gelfand - Naimark theorem can be thought of as the construction of two contravariant functors (cofunctors for short) from the category of locally compact Hausdorff spaces to the category of C^*-algebras. The first cofunctor C takes a compact space X to the C^*-algebra $C(X)$ of continuous complex-valued functions on X, and takes a continuous map $f : X \to Y$ to its transpose $C(f) : C(Y) \to C(X)$ defined by following way: $C(f) = (h \mapsto hf); (h \in C(Y))$.
2. Otherwise there exists inverse functor M that sets to any commutative C^*-algebra A space of its characters $M(A)$. Many topological results related to locally compact spaces has its (noncommutative) algebraic analogues.

The Noncommutative geometry is THE POINT IS THAT THERE IS NO POINT. Noncommutative C^*-algebra is being considered as noncommutative generalization of locally compact Hausdorff space.

TOPOLOGY	ALGEBRA
Locally compact space	C^* - algebra
Compact space	Unital C^* - algebra
Continuous map	*-homomorpfism
Minimal compactification	Unitization
Maximal compactification	Algebra if multpicators
Closed subset	Ideal
Morphism of covering	?
Pointed space (X, x_0)	?
Fundamental group	?
Singular homology	?
Hurewicz homomorphism	?

Noncommutative geometry contains MORE QUESTIONS THEN ANSWERS

The Noncommutative geometry is THE POINT IS THAT THERE IS NO POINT. Noncommutative C^*-algebra is being considered as noncommutative generalization of locally compact Hausdorff space.

TOPOLOGY	ALGEBRA
Locally compact space	C^* - algebra
Compact space	Unital C^* - algebra
Continuous map	*-homomorpfism
Minimal compactification	Unitization
Maximal compactification	Algebra if multpicators
Closed subset	Ideal
Morphism of covering	?
Pointed space (X, x_0)	?
Fundamental group	?
Singular homology	?
Hurewicz homomorphism	?

Noncommutative geometry contains MORE QUESTIONS THEN ANSWERS

Main questions

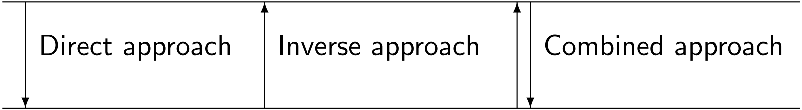
1. What is analogue of $H_1(X)$?
2. What is analogue of pointed space (X, x_0)?
3. What is analogue of $\pi_1(X, x_0)$?
4. What is analogue of Hurewicz homomorphism?

There is a set of versions of answers which depend on context. Analogue of $H_1(X)$ for Hurewicz theorem can be different from analogue of $H_1(X)$ for other problems.

There are three approaches for solution of the problem:

1. Direct (Deductive) From analogues of definitions to analogues of theorems;
2. Inverse From analogues of theorems to analogues of definitions;
3. Combined Simultaneous development of analogues of definitions and theorems.

Fundamental group notion



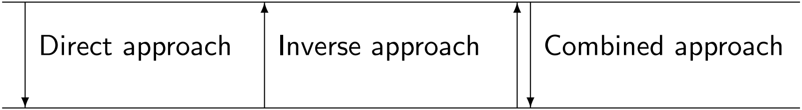
Theorems which are based on "fundamental group" notion

One of theorem is analogue of Hurewicz theorem. Since Hurewicz theorem contains information about fundamental group the analogue of Hurewicz theorem could help to state definition of fundamental group.

There are three approaches for solution of the problem:

1. Direct (Deductive) From analogues of definitions to analogues of theorems;
2. Inverse From analogues of theorems to analogues of definitions;
3. Combined Simultaneous development of analogues of definitions and theorems.

Fundamental group notion



Theorems which are based on "fundamental group" notion

One of theorem is analogue of Hurewicz theorem. Since Hurewicz theorem contains information about fundamental group the analogue of Hurewicz theorem could help to state definition of fundamental group.

1. Let P be homotopy invariant covariant functor from Hausdorff locally compact topological spaces to Abelian groups which satisfy following condition $P(S^1) \sim \mathbb{Z}$;
   Then we can construct generalize of Hurewicz homomorphism replacing H_1 by P and construct analogue of Hurewicz homomorphism.
   $\phi : \pi_1(X, x_0) \to P(X)$;

2. Construction
   Let c be generator of Abelian group $P(S^1)$ and $f : (S, s_0) \to X$ continuous map which represent element $[f] \in \pi_1(X, x_0)$. We suppose that analogue of Hurewicz homomorphism ϕ is defined by following way:
   $\phi([f]) = P(f)(c)$;

1. Let P be homotopy invariant covariant functor from Hausdorff locally compact topological spaces to Abelian groups which satisfy following condition $P(S^1) \sim \mathbb{Z}$;
   Then we can construct generalize of Hurewicz homomorphism replacing H_1 by P and construct analogue of Hurewicz homomorphism.
   $\phi : \pi_1(X, x_0) \to P(X)$;

2. Construction
   Let c be generator of Abelian group $P(S^1)$ and $f : (S, s_0) \to X$ continuous map which represent element $[f] \in \pi_1(X, x_0)$. We suppose that analogue of Hurewicz homomorphism ϕ is defined by following way:
   $\phi([f]) = P(f)(c)$;

1. Let P be homotopy invariant covariant functor from Hausdorff locally compact topological spaces to Abelian groups which satisfy following condition $P(S^1) \sim \mathbb{Z}$; M is contravariant functor from commutative C^* algebras to Hausdorff locally compact topological spaces which sets for C^* commutative algebra a space of its characters.
2. Then PM is a contravariant homotopy invariant functor from from commutative C^* - algebras to Abelian groups which satisfy condition $PM(C(S^1)) \sim \mathbb{Z}$
So noncommutative analogue of P is a contravariant homotopy invariant functor R from from (sub)category (noncommutative) C^* - algebras to Abelian groups which satisfy condition $R(C(S^1)) \sim \mathbb{Z}$
3. Example Let K^1 be functor of K homology. Then $K^1(C(S^1)) = \mathbb{Z}$. In this article the $K^1(A)$ as analogue of H_1 is being considered.

1. Let P be homotopy invariant covariant functor from Hausdorff locally compact topological spaces to Abelian groups which satisfy following condition $P(S^1) \sim \mathbb{Z}$; M is contravariant functor from commutative C^* algebras to Hausdorff locally compact topological spaces which sets for C^* commutative algebra a space of its characters.
2. Then PM is a contravariant homotopy invariant functor from from commutative C^* - algebras to Abelian groups which satisfy condition $PM(C(S^1)) \sim \mathbb{Z}$

   So noncommutative analogue of P is a contravariant homotopy invariant functor R from from (sub)category (noncommutative) C^* - algebras to Abelian groups which satisfy condition $R(C(S^1)) \sim \mathbb{Z}$
3. Example Let K^1 be functor of K homology. Then $K^1(C(S^1)) = \mathbb{Z}$. In this article the $K^1(A)$ as analogue of H_1 is being considered.

1. Let P be homotopy invariant covariant functor from Hausdorff locally compact topological spaces to Abelian groups which satisfy following condition $P(S^1) \sim \mathbb{Z}$; M is contravariant functor from commutative C^* algebras to Hausdorff locally compact topological spaces which sets for C^* commutative algebra a space of its characters.
2. Then PM is a contravariant homotopy invariant functor from from commutative C^* - algebras to Abelian groups which satisfy condition $PM(C(S^1)) \sim \mathbb{Z}$
   So noncommutative analogue of P is a contravariant homotopy invariant functor R from from (sub)category (noncommutative) C^* - algebras to Abelian groups which satisfy condition $R(C(S^1)) \sim \mathbb{Z}$
3. Example Let K^1 be functor of K homology. Then $K^1(C(S^1)) = \mathbb{Z}$. In this article the $K^1(A)$ as analogue of H_1 is being considered.

1. Let P be homotopy invariant covariant functor from Hausdorff locally compact topological spaces to Abelian groups which satisfy following condition $P(S^1) \sim \mathbb{Z}$; M is contravariant functor from commutative C^* algebras to Hausdorff locally compact topological spaces which sets for C^* commutative algebra a space of its characters.
2. Then PM is a contravariant homotopy invariant functor from from commutative C^* - algebras to Abelian groups which satisfy condition $PM(C(S^1)) \sim \mathbb{Z}$
   So noncommutative analogue of P is a contravariant homotopy invariant functor R from from (sub)category (noncommutative) C^* - algebras to Abelian groups which satisfy condition $R(C(S^1)) \sim \mathbb{Z}$
3. Example Let K^1 be functor of K homology. Then $K^1(C(S^1)) = \mathbb{Z}$. In this article the $K^1(A)$ as analogue of H_1 is being considered.

Analogue of connected component

1. If $X = \coprod_i X_i$ and all X_i are connected then all algebras $C(X_i)$ are simple and $C(X) = \bigoplus_i C(X_i)$
2. If C^* - algebra $A = \bigoplus_i A_i$ then i - th connected component is associated to A_i (A_i is simple algebra.
3. If A is unital then $1_A = \sum_i 1_{A_i}$ and 1_{A_i} is selfadjoint idempotent of A.

Noncommutative analogue of pointed space (X, x_0)

1. Any point $x_0 \in X$ defines homomorphism $H_0(\{x_0\}) \to H_0(X)$ and generator of $h \in H_0(X)$. If $X = \coprod_i X_i$ then $H_0(X) = \bigoplus_i H_0(X_i)$ and $H^0(X) = \bigoplus_i H^0(X_i)$, $H_0(X) \sim H^0(X_i) \sim \mathbb{Z}$. Generator h defines path-connected component of x_0 satisfies following conditions:
   1. h has infinite period
   2. h is not divisible
   3. If h_{i_0} generator of $H_0(X_{i_0})$ and h^i generator of $H^0(X_{i_0})$ then $h^{i_0} \frown h_{i_0} = h_{i_0}$ and $h^i \frown h_{i_0} = 0$ $(i_0 \neq i)$.
2. Since we consider K^1 as analogue of H_1 then it is reasonable consider K^0 as analogue of H_0. If $h' \in K^0(A)$ is analogue of $h \in X$ then we can define following requirements.
   1. h' has infinite period
   2. h' is not divisible
   3. if $A = \bigoplus_i A_i$ and $1_A = \sum_i 1_{A_i}$ then there exits such single index i_0 that $h' \cdot [1_{A_{i_0}}] = 1_{KK_0(\mathbb{C},\mathbb{C})}$ and $h' \cdot [1_{A_i}] = 0$.

Pair (A, h') will be called analogue of pointed space.

Noncommutative analogue of pointed space (X, x_0)

1. Any point $x_0 \in X$ defines homomorphism $H_0(\{x_0\}) \to H_0(X)$ and generator of $h \in H_0(X)$. If $X = \coprod_i X_i$ then $H_0(X) = \bigoplus_i H_0(X_i)$ and $H^0(X) = \bigoplus_i H^0(X_i)$, $H_0(X) \sim H^0(X_i) \sim \mathbb{Z}$. Generator h defines path-connected component of x_0 satisfies following conditions:
   1. h has infinite period
   2. h is not divisible
   3. If h_{i_0} generator of $H_0(X_{i_0})$ and h^i generator of $H^0(X_{i_0})$ then $h^{i_0} \frown h_{i_0} = h_{i_0}$ and $h^i \frown h_{i_0} = 0$ $(i_0 \neq i)$.
2. Since we consider K^1 as analogue of H_1 then it is reasonable consider K^0 as analogue of H_0. If $h' \in K^0(A)$ is analogue of $h \in X$ then we can define following requirements.
   1. h' has infinite period
   2. h' is not divisible
   3. if $A = \bigoplus_i A_i$ and $1_A = \sum_i 1_{A_i}$ then there exits such single index i_0 that $h' \cdot [1_{A_{i_0}}] = 1_{KK_0(\mathbb{C},\mathbb{C})}$ and $h' \cdot [1_{A_i}] = 0$.

Pair (A, h') will be called analogue of pointed space.

Main requirements to analogues of fundamental group and Hurewicz homomorphism

1. Analogue of fundamental group
   1. Definition Noncommutative analogue of fundamental group is a map which sets to any analogue of pointed space (A, h) group $\pi_1(A, h)$.
   2. Requirement If X is locally compact Hausdorff space then $\pi_1(C(X), h) \sim \pi_1(X, x_0)$. If this requirement is satisfied then fundamental group $\pi_1(A, h)$ is called "good".
2. Analogue of Hurewicz homomorphism
   1. Definition Noncommutative analogue of Hurewicz homomorphism is homomorphism from $\pi_1(A, h)$ to $K^1(A)$.
   2. Requirement If X is locally compact Hausdorff space analogue of Hurewicz homomorphism corresponds to natural homomorphism $f : \pi_1(X, x_0) \to K^1(C(X))$

Main requirements to analogues of fundamental group and Hurewicz homomorphism

1. Analogue of fundamental group
   1. Definition Noncommutative analogue of fundamental group is a map which sets to any analogue of pointed space (A, h) group $\pi_1(A, h)$.
   2. Requirement If X is locally compact Hausdorff space then $\pi_1(C(X), h) \sim \pi_1(X, x_0)$. If this requirement is satisfied then fundamental group $\pi_1(A, h)$ is called "good".
2. Analogue of Hurewicz homomorphism
   1. Definition Noncommutative analogue of Hurewicz homomorphism is homomorphism from $\pi_1(A, h)$ to $K^1(A)$.
   2. Requirement If X is locally compact Hausdorff space analogue of Hurewicz homomorphism corresponds to natural homomorphism $f : \pi_1(X, x_0) \to K^1(C(X))$

Different definitions of fundamental group

1. Definition 1 Let X be topological space and $x_0 \in X$ is its point. Then fundamental group $\pi_1(X, x_0)$ as a set is a set of homotopy classes $[S^1, s_0; X, x_0]$. Since the noncommutative geometry is THE POINT IS THAT THERE IS NO POINT this definition is not suitable.
2. Definition 2 Fundamental group is a group $G(\tilde{X}|X)$ of covering transformations of universal covering $\tilde{X}$ of X.
3. Definition 1 does not have good noncommutative generalization. We need noncommutative analogue of $\tilde{X}$ for Definition 2 generalization. This problem is only partially solved.

Different definitions of fundamental group

1. Definition 1 Let X be topological space and $x_0 \in X$ is its point. Then fundamental group $\pi_1(X, x_0)$ as a set is a set of homotopy classes $[S^1, s_0; X, x_0]$. Since the noncommutative geometry is THE POINT IS THAT THERE IS NO POINT this definition is not suitable.
2. Definition 2 Fundamental group is a group $G(\tilde{X}|X)$ of covering transformations of universal covering $\tilde{X}$ of X.
3. Definition 1 does not have good noncommutative generalization. We need noncommutative analogue of $\tilde{X}$ for Definition 2 generalization. This problem is only partially solved.

Different definitions of fundamental group

1. Definition 1 Let X be topological space and $x_0 \in X$ is its point. Then fundamental group $\pi_1(X, x_0)$ as a set is a set of homotopy classes $[S^1, s_0; X, x_0]$. Since the noncommutative geometry is THE POINT IS THAT THERE IS NO POINT this definition is not suitable.
2. Definition 2 Fundamental group is a group $G(\tilde{X}|X)$ of covering transformations of universal covering $\tilde{X}$ of X.
3. Definition 1 does not have good noncommutative generalization. We need noncommutative analogue of $\tilde{X}$ for Definition 2 generalization. This problem is only partially solved.

Analogue of $\tilde{X}$

1. Universal covering $\tilde{X}$ is universal (maximal) in category of coverings of X.
2. There are following approaches of definition of maximal covering object:
    1. Define good analogue of category of coverings and looking for its universal object;
    2. Define good analogue of covering objects and partial order on these objects. Then looking for maximal object and proving its unique property.

1. Definition (Miyashita 1966) Let $f : A \to B$ be homomorphism of algebras and G is finite group of automorphisms of A. Let $h : A \otimes_B A \to Map(G, A)$ is a map defined by following way: $a_1 \otimes a_2 \mapsto (g \mapsto a_1 \otimes ga_2)$. Homomorphism f is called G - Galois if following two conditions are satisfied
   1. $A = B^G$; (G is denoted by $G(B|A)$).
   2. Map h is bijective.
2. If A and B is commutative then $M(B) \to M(A)$ is finitely listed covering.
3. If $f : A \to B$ is G - Galois extension then there is natural isomorphism $K^*(A) \sim K^*_G(B)$

This definition provides good generalization of finitely listed covering. So we call *- homomorphism f finitely listed covering if f finite G - Galois extension.

1. Definition (Miyashita 1966) Let $f : A \to B$ be homomorphism of algebras and G is finite group of automorphisms of A. Let $h : A \otimes_B A \to Map(G, A)$ is a map defined by following way: $a_1 \otimes a_2 \mapsto (g \mapsto a_1 \otimes ga_2)$. Homomorphism f is called G - Galois if following two conditions are satisfied
   1. $A = B^G$; (G is denoted by $G(B|A)$).
   2. Map h is bijective.
2. If A and B is commutative then $M(B) \to M(A)$ is finitely listed covering.
3. If $f : A \to B$ is G - Galois extension then there is natural isomorphism $K^*(A) \sim K^*_G(B)$

This definition provides good generalization of finitely listed covering. So we call *- homomorphism f finitely listed covering if f finite G - Galois extension.

1. Definition (Miyashita 1966) Let $f : A \to B$ be homomorphism of algebras and G is finite group of automorphisms of A. Let $h : A \otimes_B A \to Map(G, A)$ is a map defined by following way: $a_1 \otimes a_2 \mapsto (g \mapsto a_1 \otimes ga_2)$. Homomorphism f is called G - Galois if following two conditions are satisfied
   1. $A = B^G$; (G is denoted by $G(B|A)$).
   2. Map h is bijective.
2. If A and B is commutative then $M(B) \to M(A)$ is finitely listed covering.
3. If $f : A \to B$ is G - Galois extension then there is natural isomorphism $K^*(A) \sim K^*_G(B)$

This definition provides good generalization of finitely listed covering. So we call *- homomorphism f finitely listed covering if f finite G - Galois extension.

1. Besides finite coverings we should have analogue of infinite coverings.
2. Definition (Miyashita 1967) Let G be discrete group. Homomorphism $f : A \to B$ is called locally finite G - Galois extension if there are fixed normal subgroups N_λ $(\lambda \in \Lambda)$ which satisfy the following conditions:
   1. $(G : N_\lambda) < \infty$ and $A \to B^{N_\lambda}$ is G/N_λ - Galois extension;
   2. $B = \cup_\lambda B^{N_\lambda}$, and $\{B^{N_\lambda} : \lambda \in \Lambda\}$ is a directed set with respect to inclusion relation ($\cup_\lambda B^{N_\lambda}$ is directed union).
3. This definition do not provide good generalization of finitely listed covering by following reasons:
   1. Let X be compact Hausdorff space. Suppose that $Y \to X$ infinitely listed covering and $G = G(Y|X)$ is infinite covering group;
   2. Then Y is not compact and $C_0(Y)$ is not unital;
   3. If A is unital algebra then every locally finite G - Galois extension of A is unital.

1. Besides finite coverings we should have analogue of infinite coverings.
2. Definition (Miyashita 1967) Let G be discrete group. Homomorphism $f : A \to B$ is called locally finite G - Galois extension if there are fixed normal subgroups N_λ $(\lambda \in \Lambda)$ which satisfy the following conditions:
   1. $(G : N_\lambda) < \infty$ and $A \to B^{N_\lambda}$ is G/N_λ - Galois extension;
   2. $B = \cup_\lambda B^{N_\lambda}$, and $\{B^{N_\lambda} : \lambda \in \Lambda\}$ is a directed set with respect to inclusion relation ($\cup_\lambda B^{N_\lambda}$ is directed union).
3. This definition do not provide good generalization of finitely listed covering by following reasons:
   1. Let X be compact Hausdorff space. Suppose that $Y \to X$ infinitely listed covering and $G = G(Y|X)$ is infinite covering group;
   2. Then Y is not compact and $C_0(Y)$ is not unital;
   3. If A is unital algebra then every locally finite G - Galois extension of A is unital.

Noncommutative analogues of infinite coverings.

1. Sketch of definition
   1. Let A, B be C^* algebras and $M(B)$ is algebra of multipliers of B.
   2. Locally finite covering is locally finite G - Galois *-homomorphism.
2. Example
3. Let A be C^* - algebra and $u \in A$ such unitary element that $[u] \in K_1(A)$ is nontrivial generator of infinite order.
4. It is (not unique) sequence of C^* - algebras $A \subset A[v_1] \subset A[v_2] \subset ...$ which match following requirements: $v_n^n = u$ $(n \in \mathbb{N})$. $A(v_n)$ is $\mathbb{Z}_n$ - Galois extension.
5. It is evident left action of $C(u)$ on A. Let us define action of $C[u]$ on $C_0(\mathbb{R}$ by following way: $uf = e^{2\pi i x} f$ $(f \in C_0(\mathbb{R})$, $x \in \mathbb{R})$.
6. There are such algebra B that $B \sim A \otimes_{C(u)} \mathbb{R}$ as $A[v_n] - \mathbb{R}$ as bimodules and there are * - homomorphisms $A(v_n) \to M(B)$.

Let A be commutative C^* - algebra generated by two unitary elements u and v:
$uu^* = u^*u = vv^* = v^*v = 1$; $uv = vu$.

1. Let B' be C^* - algebra generated by unitary elements x', y': $x'x'^* = x'^*x' = y'y'^* = y'^*y' = 1$; $x'y' = y'x'$.
2. Let $f' : A \to B'$ $\mathbb{Z}_2$ be Galois $*-$ homomorphism defined by following way:
   $u \mapsto x'^2$; $v \mapsto y'$.
3. Let B' be C^* - algebra generated by unitary elements x', y': $x''x''^* = x''^*x'' = y''y''^* = y''^*y'' = 1$; $x''y'' = -y''x''$.
4. Let $f' : A \to B'$ $\mathbb{Z}_2$ be Galois $*-$ homomorphism defined by following way:
   $u \mapsto x''^2$; $v \mapsto y''$.

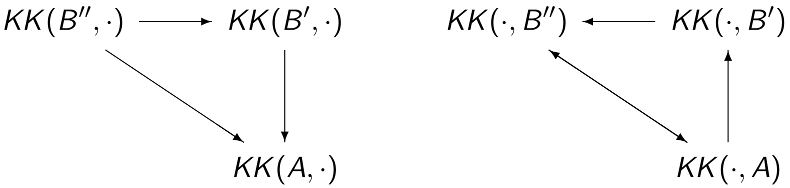
Morphisms f' and f'' should be equivalent for good analogue of fundamental group. But $B' \nsim B''$. So * - homomorphisms are not good covering morphisms.

Definition $f' : A \to B'$ and $f'' : A \to B''$ two coverings and G', G'' are their groups. Morphism of coverings is a pair $(\phi, {}_{B'}H_{B''})$ of surjective group homomorphism $\phi : G' \to G''$ and $B' - B''$ bimodule ${}_{B'}H_{B''}$ which satisfies following conditions and KK compatibility axiom:

1. G' acts on ${}_{B'}H_{B''}$ this action makes ${}_{B'}H_{B''}$ equivariant left B' - module. This action and homomorphism ϕ makes ${}_{B'}H_{B''}$ equivariant right B'' module;
2. $B' \sim End_{B''}({}_{B'}H_{B''})$ and $B'' \sim End_{B'}({}_{B'}H_{B''})^{G'/G''}$

Homomorphisms $f' : A \to B'$ and $f'' : A \to B''$ induce natural homomorphisms $KK_*(\cdot, A) \to KK_*(\cdot, B')$, $KK_*(\cdot, A) \to KK(\cdot, B'')$, $KK(B', \cdot) \to KK(A, \cdot)$, $KK(B', \cdot) \to KK(A, \cdot)$. Bimodule ${}_{B'}H_{B''}$ induces natural homomorphisms $KK(\cdot, B') \to KK(\cdot, B'')$ and $KK(B'', \cdot) \to KK(B'', \cdot)$.

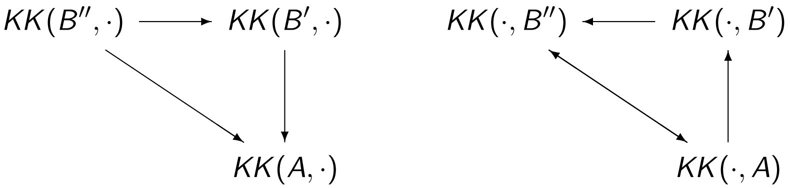
KK Compatibility axiom. Following diagrams should be commutative



Question For which C^* algebras universal object of coverings category is exist?

Homomorphisms $f' : A \to B'$ and $f'' : A \to B''$ induce natural homomorphisms $KK_*(\cdot, A) \to KK_*(\cdot, B')$, $KK_*(\cdot, A) \to KK(\cdot, B'')$, $KK(B', \cdot) \to KK(A, \cdot)$, $KK(B', \cdot) \to KK(A, \cdot)$. Bimodule ${}_{B'}H_{B''}$ induces natural homomorphisms $KK(\cdot, B') \to KK(\cdot, B'')$ and $KK(B'', \cdot) \to KK(B'', \cdot)$.

KK Compatibility axiom. Following diagrams should be commutative



Question For which C^* algebras universal object of coverings category is exist?

1. Let homomorphisms $f' : A \to B'$ and $f'' : A \to B''$ are coverings. If C is C^*- algebra than these coverings induce natural homomorphisms $f_C'^* : KK_*(C, B') \to KK_*(C, A)$ and $f_C''^* : KK_*(C, B'') \to KK_*(C, A)$.
2. Definition We say that f' is greater or equal than f'' ($f' \geq f''$)if for any C^*- algebra C following condition is satisfied $imf_C'^* \subseteq f_C''^*$.
3. Example Let X, X', X'' be locally compact Hausdorff topological spaces and $f : X' \to X$, $g : X'' \to X$ are coverings. Then $fg : X'' \to X$ is covering and there are *- homomorphisms $C(f) : C(X') \to C(X)$ and $C(fg) : C(X'') \to C(X)$. In this case we have $C(fg) \geq C(f)$.
4. Question For which C^* algebras do maximal covering exists?

1. Let homomorphisms $f' : A \to B'$ and $f'' : A \to B''$ are coverings. If C is C^*- algebra than these coverings induce natural homomorphisms $f'^*_C : KK_*(C, B') \to KK_*(C, A)$ and $f''^*_C : KK_*(C, B'') \to KK_*(C, A)$.
2. Definition We say that f' is greater or equal than f'' ($f' \geq f''$)if for any C^*- algebra C following condition is satisfied $imf'^*_C \subseteq f''^*_C$.
3. Example Let X, X', X'' be locally compact Hausdorff topological spaces and $f : X' \to X$, $g : X'' \to X$ are coverings. Then $fg : X'' \to X$ is covering and there are *- homomorphisms $C(f) : C(X') \to C(X)$ and $C(fg) : C(X'') \to C(X)$. In this case we have $C(fg) \geq C(f)$.
4. Question For which C^* algebras do maximal covering exists?

1. Let homomorphisms $f' : A \to B'$ and $f'' : A \to B''$ are coverings. If C is C^*- algebra than these coverings induce natural homomorphisms $f'^*_C : KK_*(C, B') \to KK_*(C, A)$ and $f''^*_C : KK_*(C, B'') \to KK_*(C, A)$.
2. Definition We say that f' is greater or equal than f'' ($f' \geq f''$)if for any C^*- algebra C following condition is satisfied $imf'^*_C \subseteq f''^*_C$.
3. Example Let X, X', X'' be locally compact Hausdorff topological spaces and $f : X' \to X$, $g : X'' \to X$ are coverings. Then $fg : X'' \to X$ is covering and there are *- homomorphisms $C(f) : C(X') \to C(X)$ and $C(fg) : C(X'') \to C(X)$. In this case we have $C(fg) \geq C(f)$.
4. Question For which C^* algebras do maximal covering exists?

1. Hurewicz homomorphism.
   1. Analogue of universal covering $\tilde{X}$ is not yet defined;
   2. We cannot define analogue of $\pi_1(X, x_0)$.
   3. So cannot define analogue analogue of Hurewicz homomorphism.
2. Hurewicz homomorphism associated with covering.
   1. Analogue of covering is defined;
   2. Analogue of covering group $G(Y|X)$ is $G(B|A)$;
   3. Let us define homomorphism $G(B|A) \to K^1(A)$.

1. Definition
Abelian fundamental group $\pi_{ab}(X)$ is defined by following equation.
$\pi_{ab}(X) = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$;
Note: Abelian fundamental group does not depend on x_0.
2. It is one to one correspondence between homomorphisms from $\pi_1(X)$ to Abelian group A and homomorphisms from $\pi_{ab}(X)$ to A. Definition of $\pi_{ab}(X)$ could be easy then definition of $\pi_1(X)$
3. Since $K^1(A)$ is Abelian group then Hurewicz homomorphism $\pi_1(A)) \to K^1(A)$ could be decomposed by following way $\pi_1(A) \to \pi_{ab}(A) \to K^1(A)$. So we need π_{ab} only for definition of Hurewicz homomorphism.

1. Definition
   Abelian fundamental group $\pi_{ab}(X)$ is defined by following equation.
   $\pi_{ab}(X) = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$;
   Note: Abelian fundamental group does not depend on x_0.
2. It is one to one correspondence between homomorphisms from $\pi_1(X)$ to Abelian group A and homomorphisms from $\pi_{ab}(X)$ to A. Definition of $\pi_{ab}(X)$ could be easy then definition of $\pi_1(X)$
3. Since $K^1(A)$ is Abelian group then Hurewicz homomorphism $\pi_1(A)) \to K^1(A)$ could be decomposed by following way $\pi_1(A) \to \pi_{ab}(A) \to K^1(A)$. So we need π_{ab} only for definition of Hurewicz homomorphism.

1. **Definition**
   Abelian fundamental group $\pi_{ab}(X)$ is defined by following equation.
   $\pi_{ab}(X) = \pi_1(X, x_0)/[\pi_1(X, x_0), \pi_1(X, x_0)]$;
   Note: Abelian fundamental group does not depend on x_0.
2. It is one to one correspondence between homomorphisms from $\pi_1(X)$ to Abelian group A and homomorphisms from $\pi_{ab}(X)$ to A. Definition of $\pi_{ab}(X)$ could be easy then definition of $\pi_1(X)$
3. Since $K^1(A)$ is Abelian group then Hurewicz homomorphism $\pi_1(A)) \to K^1(A)$ could be decomposed by following way $\pi_1(A) \to \pi_{ab}(A) \to K^1(A)$. So we need π_{ab} only for definition of Hurewicz homomorphism.

1. The universal covering is maximal among all coverings.
2. Since Hurewicz homomorphism depends on $\pi_{ab}(X)$ only class of coverings could be restricted to Abelian coverings (coverings with Abelian covering group).
3. Maximal Abelian covering is denoted by X_{ab}.

Class field theory considers Abelian extensions of fields only and provides particular calculation of Galois group. It is following analogy.

ALGEBRAIC TOPOLOGY	CLASS FIELD THEORY
Topological space X	Field k
Universal covering space $\tilde{X}$	Algebraic closure $\bar{k}$
Maximal Abelian covering X_{ab}	k_{ab}
$\tilde{X} \to X_{ab} \to X$	$k \in k_{ab} \in \bar{k}$
$\pi_1(X) = G(\tilde{X}\|X)$	$Gal(\bar{k}, k)$
$\pi_{ab}(X) = G(X_{ab}\|X) \in \pi_1(X) = G(\tilde{X}\|X)$	$Gal(k_{ab}, k) \in Gal(\bar{k}, k)$

1. The universal covering is maximal among all coverings.
2. Since Hurewicz homomorphism depends on $\pi_{ab}(X)$ only class of coverings could be restricted to Abelian coverings (coverings with Abelian covering group).
3. Maximal Abelian covering is denoted by X_{ab}.

Class field theory considers Abelian extensions of fields only and provides particular calculation of Galois group. It is following analogy.

ALGEBRAIC TOPOLOGY	CLASS FIELD THEORY
Topological space X	Field k
Universal covering space $\tilde{X}$	Algebraic closure $\bar{k}$
Maximal Abelian covering X_{ab}	k_{ab}
$\tilde{X} \to X_{ab} \to X$	$k \in k_{ab} \in \bar{k}$
$\pi_1(X) = G(\tilde{X}\|X)$	$Gal(\bar{k}, k)$
$\pi_{ab}(X) = G(X_{ab}\|X) \in \pi_1(X) = G(\tilde{X}\|X)$	$Gal(k_{ab}, k) \in Gal(\bar{k}, k)$

Construction of Hurewicz homomorphism

1. Assumptions
   1. Let $A \to B$ be G - Galois extension and G is finitely generated Abelian group;
   2. Let (A, h) is analogue of pointed space.
2. Construction
   1. It is natural isomorphism: $K^*_G(B) \to K^*(A)$;
   2. It is natural isomorphism $G \sim KK^1_G(\mathbb{C}, \mathbb{C})$;
   3. $KK^1_G(\mathbb{C}, \mathbb{C})$ acts on $KK^0_G(B, \mathbb{C}) \sim K^0_G(B) \sim K^0(A)$, So G acts on $K^0(A)$, it is pairing $G \times K^0(A) \to K^1(A)$;
   4. Analogue of Hurewicz homomorphism is defined as $G \ni g \mapsto (gh - h) \in K^1(A)$.

This construction provides good Hurewicz homomorphism in particular cases.

Finite and infinite parts of Hurewicz homomorphism

Let $A \to B$ be covering with finitely generated Abelian covering group G and $Tors(G)$ is its torsion. It is following diagram:

$$
\begin{array}{ccccc}
Tors(G) & \longrightarrow & G & \longrightarrow & G/Tors(G) \\
\downarrow \phi^1_{fin} & & \downarrow \phi^1 & & \downarrow \phi^1_{inf} \\
Ext^1(KK^0_G(\mathbb{C},\mathbb{C}),\mathbb{Z}) & \longrightarrow & KK^1_G(\mathbb{C},\mathbb{C}) & \longrightarrow & Hom(KK^1_G(\mathbb{C},\mathbb{C}),\mathbb{Z}) \\
\downarrow \phi^2_{fin} & & \downarrow \phi^2 & & \downarrow \phi^2_{inf} \\
Ext^1(K_0(A),\mathbb{Z}) & \longrightarrow & K^1(A) & \longrightarrow & Hom(K_1(A),\mathbb{Z})
\end{array}
$$

Here $\phi^2\phi^1$ is Hurewicz homomorphism. Let us call $\phi_{fin} = \phi^2_{fin}\phi^1_{fin}$ ($\phi_{inf} = \phi^2_{inf}\phi^1_{inf}$) finite (infinite) part of Hurewicz homomorphism.

Finite part of Hurewicz homomorphism

1. Let $A \to B$ be G - covering and $G \sim \mathbb{Z}_n$.
2. B is finitely generated projective module. Let $r : K(A) \to K(A)$ is homomorphism defined by following way $K_0(A) \ni [P] \mapsto [P \otimes_A B] \in K_0(A)$ Suppose that $ker(r) = K \subseteq K_0(A)$ is isomorphic to $\mathbb{Z}_n$.
3. G acts on K and this action induce pairing: $G \times K \to \mathbb{Z}_n$;
4. It is subgroup $L \in K^1(A)$ that it is following covering: $L \times K \to \mathbb{Z}_n$;
5. These pairings induce homomorphism $G \to L$ which is a part of Hurewicz homomorphism $G \to L \subseteq K^1(A)$.

Infinite part of Hurewicz homomorphism

1. Suppose that $u_1, ..., u_n \in A$ such unitary elements that $[u_1], ..., [u_n] \in K_1(A)$ have infinite rank end these elements are not divisible in $K_1(A)$. Suppose that $[u_1], ..., [u_n]$ generate subgroup $\mathbb{Z}^n$.
2. It is $\mathbb{Z}^n$ covering $f : A \to B$ that $f_*([u_i]) = 0 \in K_1(B)$.
3. There are such generators $g_1, ..., g_n \in \mathbb{Z}^n \subseteq K_1(A)$ that $[u_i]\phi(g_j) \in \delta_{ij}1_{KK^0(\mathbb{C},\mathbb{C})}$.
4. Let A is commutative then:
    1. $[u_1], ..., [u_n]$ are represented by continuous maps $r_i : M(A) \to S^1$ $(i = 1, ...n)$.
    2. there are right inverse maps $s_i : S^1 \to M(A)$ of r_i $(r_i s_i = Id_{S^1})$.
    3. Hurewicz homomorphism is following map $g_i \mapsto r_i(c)$ where $c \in K^1(C(S^1))$ is generator of $K^1(C(S^1))$.

Wait following results.

Thank You!